17

Homogenization for advection-diffusion in a perforated domain

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Abstract

The volume of a Wiener sausage constructed from a diffusion process with periodic, mean-zero, divergence-free velocity field, in dimension 3 or more, is shown to have a non-random and positive asymptotic rate of growth. This is used to establish the existence of a homogenized limit for such a diffusion when subject to Dirichlet conditions on the boundaries of a sparse and independent array of obstacles. There is a constant effective long-time loss rate at the obstacles. The dependence of this rate on the form and intensity of the obstacles and on the velocity field is investigated. A Monte Carlo algorithm for the computation of the volume growth rate of the sausage is introduced and some numerical results are presented for the Taylor–Green velocity field.

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1 Introduction

We consider the problem of the existence and characterization of a homogenized limit for advection-diffusion in a perforated domain. This problem was initially motivated for us as a model for the transport of water vapour in the atmosphere, subject to molecular diffusion and turbulent advection, where the vapour is also lost by condensation on suspended ice crystals. It is of interest to determine the long-time rate of loss and in particular whether this is strongly affected by the advection. In this article we address a simple version of this set-up, where the advection is periodic in space and constant in time and where the ice crystals remain fixed in space.

Let K be a compact subset of $\mathbb{R}^d$ of positive Newtonian capacity. We assume throughout that $d \geqslant 3$. Let $\rho \in (0, \infty)$. We consider eventually the limit $\rho \to 0$. Construct a random perforated domain $D \subseteq \mathbb{R}^d$ by removing all the sets $K + p$, where p runs over the support P of a Poisson random measure μ on $\mathbb{R}^d$ of intensity ρ. Let v be a $\mathbb{Z}^d$-periodic, Lipschitz, mean-zero, divergence-free vector field on $\mathbb{R}^d$. Our aim is to determine the long-time behaviour, over times of order $\sigma^2 = \rho^{-1}$, of advection-diffusion in the domain D corresponding to the operator[1]

$$\mathcal{L} = \tfrac{1}{2}\Delta + v(x).\nabla$$

with Dirichlet boundary conditions. It is well known (see Section 2) that the long-time behaviour of advection-diffusion in the whole space $\mathbb{R}^d$ can be approximated by classical, homogeneous, heat-flow, with a constant diffusivity matrix $\bar{a} = \bar{a}(v)$. The effect of placing Dirichlet boundary conditions on the sets $K + p$ is to induce a loss of heat. The homogenization problem in a perforated domain has been considered already in the case of Brownian motion [5], [7], [12], [15] and Brownian motion with constant drift [3]. The novelty here is to explore the possible interaction between inhomogeneity in the drift and in the domain. We will show that as $\rho \to 0$ there exists an effective constant loss rate $\bar{\lambda}(v, K)$ in the time-scale σ^2. We will also identify the limiting values of $r^{2-d}\bar{\lambda}(v, rK)$ as $r \to 0$ and $r \to \infty$ and we will compute numerically this function of r for one choice of v and K.

Fix a function $f \in L^2(\mathbb{R}^d)$. Write $u = u(t, x)$ for the solution to the Cauchy problem for $\mathcal{L}$ in $[0, \infty) \times D$ with initial data f, and with

[1] All results to follow extend to the case of the operator $\frac{1}{2}\operatorname{div} a\nabla + v(x).\nabla$, where a is a constant positive-definite symmetric matrix, by a straightforward scaling transformation. We simplify the presentation by taking $a = I$. Results for the case $a = \varepsilon^2 I$ are stated in Section 7 for easy reference.

Dirichlet conditions on the boundary of D. Thus, for suitably regular K and f, u is continuous on $[0,\infty)\times D$ and on $(0,\infty)\times\bar{D}$, and is $C^{1,2}$ on $(0,\infty)\times D$; we have $u(0,x)=f(x)$ for all $x\in D$ and

$$\frac{\partial u}{\partial t}=\tfrac{1}{2}\Delta u+v(x).\nabla u \quad \text{on } (0,\infty)\times D.$$

We shall study the behaviour of u over large scales in the limit $\rho\to 0$. Our analysis will rest on the following probabilistic representation of u. Let X be a diffusion process in $\mathbb{R}^d$, independent of μ with generator $\mathcal{L}$ starting from x. Such a process can be realised by solving the stochastic differential equation

$$dX_t=dW_t+v(X_t)\,dt,\quad X_0=x \tag{1.1}$$

driven by a Brownian motion W in $\mathbb{R}^d$. Set

$$T=\inf\{t\geqslant 0: X_t\in K+P\}.$$

Then

$$u(t,x)=\mathbb{E}_x\left(f(X_t)1_{\{T>t\}}\middle|\mu\right).$$

The key step is to express the right hand side of this identity in terms of an analogue for X of the Wiener sausage. Associate to each path $\gamma\in C([0,\infty),\mathbb{R}^d)$ and to each interval $I\subseteq[0,\infty)$ a set $S_I^K(\gamma)\subseteq\mathbb{R}^d$ formed of the translates of K by γ_t as t ranges over I. Thus

$$S_I^K(\gamma)=\cup_{t\in I}(K+\gamma_t)=\{x\in\mathbb{R}^d: x-\gamma_t\in K \text{ for some } t\in I\}.$$

Write S_t^K for the random set $S_{(0,t]}^K(X)$ and write $|S_t^K|$ for the Lebesgue volume of S_t^K. We call S_t^K the *diffusion sausage* or (X,K)*-sausage* and refer to K as the *cross section.* Then $T>t$ if and only if $\mu(S_t^{\hat{K}})=0$, where $\hat{K}=\{-x:x\in K\}$. Hence

$$u(t,x)=\mathbb{E}_x\left(f(X_t)1_{\{\mu(S_t^{\hat{K}})=0\}}\middle|\mu\right)$$

and so, by Fubini, we obtain the formulae

$$\mathbb{E}(u(t,x))=\mathbb{E}_x\left(f(X_t)\exp(-\rho|S_t^{\hat{K}}|)\right) \tag{1.2}$$

and

$$\mathbb{E}(u(t,x)^2)=\mathbb{E}_x\left(f(X_t)f(Y_t)\exp(-\rho|S_t^{\hat{K}}(X)\cup S_t^{\hat{K}}(Y)|)\right) \tag{1.3}$$

where Y is an independent copy of X.

In the next section we review the homogenization theory for $\mathcal{L}$ in the whole space. Then, in Section 3 we show, as a straightforward application

of Kingman's subadditive ergodic theorem, that the sausage volume $|S_t^K|$ has almost surely an asymptotic growth rate $\gamma(v, K)$, which is non-random. In Section 4 we make some further preparatory estimates on diffusion sausages. Then in Section 5 we identify the limiting values of $r^{2-d}\gamma(v, rK)$ as $r \to 0$ and as $r \to \infty$. In Section 6, we use the formulae (1.2), (1.3) to deduce the existence of a homogenized scaling limit for the function u, and we prove a corresponding weak limit for the diffusion process X and the hitting time T. We shall see in particular that for large obstacles it is the effective diffusivity $\bar{a}$ which accounts for the loss of heat in the obstacles. On the other hand, when the obstacles are small, the loss of heat is controlled instead by the molecular diffusivity, even over scales where the diffusive motion itself is close to its homogenized limit. Some results for non-unit molecular diffusivity are recorded in Section 7. Finally, in Section 8, we describe a new Monte Carlo algorithm to compute the volume growth rate for the (X, K)-sausage, and hence the effective long-time rate of loss of heat. We present some numerical results obtained using the algorithm which interpolate between our theoretical predictions for large and small obstacles.

2 Review of homogenization for diffusion with periodic drift

There is a well known homogenization theory for $\mathcal{L}$-diffusion in the whole space $\mathbb{R}^d$. See [1], [2], [6], [11]. We review here a few basic facts which provide the background for our treatment of the case of a perforated domain. Our hypotheses on v ensure the existence of a periodic, Lipschitz, antisymmetric 2-tensor field β on $\mathbb{R}^d$ such that $\frac{1}{2}\operatorname{div}\beta = v$. So we can write $\mathcal{L}$ in the form

$$\mathcal{L} = \tfrac{1}{2}\operatorname{div}(I + \beta(x))\nabla.$$

Then $\mathcal{L}$ has a continuous heat kernel $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and there exists a constant $C < \infty$, depending only on the Lipschitz constant of v, such that, for all t, x and y,

$$C^{-1}\exp\{-C|x-y|^2/t\} \leqslant p(t, x, y) \leqslant C\exp\{-|x-y|^2/Ct\}. \tag{2.1}$$

Moreover, C may be chosen so that there also holds the following Gaussian tail estimate for the diffusion process X with generator $\mathcal{L}$ starting

from x: for all $t > 0$ and $\delta > 0$,

$$\mathbb{P}_x \left(\sup_{s \leqslant t} |X_s - x| > \delta \right) \leqslant Ce^{-\delta^2/Ct}. \tag{2.2}$$

The preceding two estimates show a qualitative equivalence between X and Brownian motion, valid on all scales. On large scales this can be refined in quantitative terms. Consider the quadratic form q on $\mathbb{R}^d$ given by

$$q(\xi) = \inf_{\theta, \chi} \int_{\mathbb{T}^d} |\xi - \operatorname{div} \chi + \beta \nabla \theta|^2 \, dx$$

where the infimum is taken over all Lipschitz functions θ and all Lipschitz antisymmetric 2-tensor fields χ on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The infimum is achieved, so there is a positive-definite symmetric matrix $\bar{a}$ such that

$$q(\xi) = \langle \xi, \bar{a}^{-1} \xi \rangle.$$

The choice $\theta = 0$ and $\chi = 0$ shows that $\bar{a} \geqslant I$. As the velocity field v is scaled up, typically it is found that $\bar{a}$ also becomes large. See for example [4] for further discussion of this phenomenon.

We state first a deterministic homogenization result. Let $f \in L^2(\mathbb{R}^d)$ and $\sigma \in (0, \infty)$ be given. Denote by u the solution to the Cauchy problem for $\mathcal{L}$ in $\mathbb{R}^d$ with initial data $f(./\sigma)$ and set $u^{(\sigma)}(t, x) = u(\sigma^2 t, \sigma x)$. Then

$$\int_{\mathbb{R}^d} |u^{(\sigma)}(t, x) - \bar{u}(t, x)|^2 \, dx \to 0 \tag{2.3}$$

as $\sigma \to \infty$, for all $t \geqslant 0$, where $\bar{u}$ is the solution to the Cauchy problem for $\frac{1}{2} \operatorname{div} \bar{a} \nabla$ in $\mathbb{R}^d$ with initial data f.

In probabilistic terms, we may fix $x \in \mathbb{R}^d$ and $\sigma \in (0, \infty)$ and consider the $\mathcal{L}$-diffusion process X starting from σx. Set $X_t^{(\sigma)} = \sigma^{-1} X_{\sigma^2 t}$. Then it is known [13] that

$$X^{(\sigma)} \to \bar{X}, \quad \text{weakly on } C([0, \infty), \mathbb{R}^d) \tag{2.4}$$

where $\bar{X}$ is a Brownian motion in $\mathbb{R}^d$ with diffusivity $\bar{a}$ starting from x. The two homogenization statements are essentially equivalent given the regularity implicit in the above qualitative estimates, the Markov property, and the identity

$$u^{(\sigma)}(t, x) = \mathbb{E}(f(X_t^{(\sigma)})).$$

3 Existence of a volume growth rate for a diffusion sausage with periodic drift

Recall that the drift v is $\mathbb{Z}^d$-periodic and divergence-free.

Theorem 3.1 *There exists a constant $\gamma = \gamma(v, K) \in (0, \infty)$ such that, for all x,*

$$\lim_{t\to\infty} \frac{|S_t^K|}{t} = \gamma, \quad \mathbb{P}_x\text{-almost surely.}$$

Proof Write π for the projection $\mathbb{R}^d \to \mathbb{T}^d$. Since v is periodic, the projected process $\pi(X)$ is a diffusion on $\mathbb{T}^d$. As v is divergence-free, the unique invariant distribution for $\pi(X)$ on $\mathbb{T}^d$ is the uniform distribution. The lower bound in (2.1) shows that the transition density of $\pi(X_1)$ on $\mathbb{T}$ is uniformly positive. By a standard argument $\pi(X)$ is therefore uniformly and geometrically ergodic. Consider the case where X_0 is chosen randomly, and independently of W, such that $\pi(X_0)$ is uniformly distributed on $\mathbb{T}^d$. Then $\pi(X)$ is stationary. For integers $0 \leqslant m < n$, define $V_{m,n} = |S^K_{(m,n]}|$. Then $V_{l,n} \leqslant V_{l,m} + V_{m,n}$ whenever $0 \leqslant l < m < n$. Since Lebesgue measure is translation invariant and $\pi(X)$ is stationary, the distribution of the array $(V_{m+k,n+k} : 0 \leqslant m < n)$ is the same for all $k \geqslant 0$. Moreover $V_{m,n}$ is integrable for all m, n by standard diffusion estimates. Hence by the subadditive ergodic theorem [8] we can conclude that, for some constant $\gamma \geqslant 0$,

$$\lim_{n\to\infty} \frac{|S_n^K|}{n} = \gamma, \quad \text{almost surely.}$$

The positivity of γ follows from the positivity of $\operatorname{cap}(K)$ using Theorem 5.1 below.

Let $\mathbb{P}_x$ be the probability measure on $C([0,\infty), \mathbb{R}^d)$ which is the law of the process X starting from x. Set

$$g(x) = \mathbb{P}_x\left(\lim_{n\to\infty} \frac{|S_n^K|}{n} = \gamma\right), \quad \tilde{g}(x) = \mathbb{P}_x\left(\lim_{n\to\infty} \frac{|S_{(1,n]}^K|}{n} = \gamma\right).$$

Then g is periodic and $\tilde{g} = g$. We have shown that

$$\int_{x\in[0,1]^d} g(x)\,dx = 1.$$

Hence $g(x) = 1$ for Lebesgue-almost-all x. But then by the Markov property, for every x,

$$g(x) = \tilde{g}(x) = \int_{\mathbb{R}^d} p(1, x, y) g(y)\,dy = 1$$

which is the desired almost-sure convergence for discrete parameter n. An obvious monotonicity argument extends this to the continuous parameter t. □

4 Estimates for the diffusion sausage

We prepare some estimates on the diffusion sausage which will be needed later. These are of a type well known for Brownian motion [9] and extend in a straightforward way using the qualitative Gaussian bounds (2.1) and (2.2).

Lemma 4.1 *For all $p \in [1, \infty)$ there is a constant $C(p, v, K) < \infty$ such that, for all $t \geqslant 0$ and all $x \in \mathbb{R}^d$,*

$$\mathbb{E}_x \left(|S_t^K|^p \right)^{1/p} \leqslant C(t+1).$$

Proof Reduce to the case $t = 1$ by subadditivity of volume and L^p-norms and by the Markov property. The estimate then follows from (2.2) since

$$|S_1^K| \leqslant \omega_d \left(\mathrm{rad}(K) + \sup_{t \leqslant 1} |X_t - x| \right)^d.$$

□

Lemma 4.2 *There is a constant $C(v, K) < \infty$ with the following property. Let X and Y be independent $\mathcal{L}$-diffusions starting from x. For all $t \geqslant 1$ and all $x \in \mathbb{R}^d$, for all $a, b \geqslant 0$,*

$$\mathbb{P}_x \left(S^K_{(at,(a+1)t]}(X) \cap S^K_{(bt,(b+1)t]}(Y) \neq \emptyset \right) \leqslant C(a+b)^{-d/2} \tag{4.1}$$

and, when $b \geqslant a + 1$,

$$\mathbb{P}_x \left(S^K_{(at,(a+1)t]}(X) \cap S^K_{(bt,(b+1)t]}(X) \neq \emptyset \right) \leqslant C(b-a-1)^{-d/2}. \tag{4.2}$$

Proof We write the proof for the case $t = 1$. The same argument applies generally. There is alternatively a reduction to the case $t = 1$ by scaling. Assume that $b \geqslant a + 1$. Write $\mathcal{F}_t$ for the σ-algebra generated by $(X_s : 0 \leqslant s \leqslant t)$ and set

$$R_a = \sup_{a \leqslant t \leqslant a+1} |X_t - X_{a+1}|, \quad R_b = \sup_{b \leqslant t \leqslant b+1} |X_t - X_b|, \quad Z = X_b - X_{a+1}.$$

Then, by (2.2),

$$\mathbb{P}_x(R_b \geqslant |Z|/3 | \mathcal{F}_b) \leqslant C e^{-|Z|^2/9C}$$

so, using (2.1),

$$\mathbb{P}_x(R_b \geqslant |Z|/3) \leqslant C\mathbb{E}_x(e^{-|Z|^2/9C}) \leqslant C(b-a-1)^{d/2}.$$

On the other hand, by (2.1) again,

$$\mathbb{P}_x(R_a \geqslant |Z|/3|\mathcal{F}_{a+1}) \leqslant C(b-a-1)^{d/2}R_a^d$$

so, using (2.2),

$$\mathbb{P}_x(R_a \geqslant |Z|/3) \leqslant C(b-a-1)^{d/2}.$$

Moreover (2.1) gives also

$$\mathbb{P}_x(2\,\mathrm{rad}(K) \geqslant |Z|/3) \leqslant C(b-a-1)^{d/2}.$$

Now if $S^K_{(a,a+1]}(X) \cap S^K_{(b,b+1]}(X) \neq \emptyset$ then either $R_a \geqslant |Z|/3$ or $R_b \geqslant |Z|/3$ or $2\,\mathrm{rad}(K) \geqslant |Z|/3$. Hence the preceding estimates imply (4.2). The proof of (4.1) is similar, resting on the fact that $X_a - Y_b$ has density bounded by $C(a+b)^{-d/2}$, and is left to the reader. □

Lemma 4.3 *As $t \to \infty$, we have*

$$\sup_x \mathbb{E}_x\left(\left|\frac{|S^K_t|}{t} - \gamma\right|\right) \to 0$$

and

$$\sup_x \mathbb{E}_x\left(\frac{|S^K_t(X) \cap S^K_t(Y)|}{t}\right) \to 0$$

where Y is an independent copy of X.

Proof Note that

$$|S^K_{(1,t+1]}| \leqslant |S^K_{t+1}| \leqslant |S^K_1| + |S^K_{(1,t+1]}|.$$

Given Lemma 4.1, the first assertion will follow if we can show that, as $t \to \infty$,

$$\sup_x \mathbb{E}_x\left(\left|\frac{|S^K_{(1,t+1]}|}{t} - \gamma\right|\right) \to 0.$$

But by the Markov property and using (2.1),

$$\begin{aligned}\mathbb{E}_x\left(\left|\frac{|S^K_{(1,t+1]}|}{t} - \gamma\right|\right) &= \int_{\mathbb{R}^d} p(1,x,y)\mathbb{E}_y\left(\left|\frac{|S^K_t|}{t} - \gamma\right|\right)\,dy \\ &\leqslant C\int_{[0,1]^d} \mathbb{E}_y\left(\left|\frac{|S^K_t|}{t} - \gamma\right|\right)\,dy \to 0\end{aligned}$$

as $t \to \infty$, where we used the almost-sure convergence $|S_t^K|/t \to \gamma$ when $\pi(X_0)$ is uniform, together with uniform integrability from Lemma 4.1 to get the final limit.

For the second assertion, choose $q \in (1, 3/2)$ and $p \in (3, \infty)$ with $1/p + 1/q = 1$. Then, for $j, k \geqslant 0$, by Lemmas 4.1 and 4.2, there is a constant $C(p, v, K) < \infty$ such that

$$\begin{aligned}
&\mathbb{E}_x(|S^K_{(j,j+1]}(X) \cap S^K_{(k,k+1]}(Y)|) \\
&\quad \leqslant \mathbb{E}_x\left(|S^K_{(j,j+1]}(X)| 1_{\{S^K_{(j,j+1]}(X) \cap S^K_{(k,k+1]}(Y) \neq \emptyset\}}\right) \\
&\quad \leqslant \mathbb{E}_x\left(|S^K_1(X)|^p\right)^{1/p} \mathbb{P}_x\left(S^K_{(j,j+1]}(X) \cap S^K_{(k,k+1]}(Y) \neq \emptyset\right)^{1/q} \\
&\quad \leqslant C(j+k)^{-d/2q}.
\end{aligned}$$

So, as $n \to \infty$, we have

$$\begin{aligned}
&\mathbb{E}_x\left(\frac{|S^K_n(X) \cap S^K_n(Y)|}{n}\right) \\
&\quad \leqslant \frac{\mathbb{E}_x(|S^K_1(X)|)}{n} + \sum_{j=1}^{n-1} \sum_{k=0}^{n-1} \frac{\mathbb{E}_x(|S^K_{(j,j+1]}(X) \cap S^K_{(k,k+1]}(Y)|)}{n} \\
&\quad \leqslant Cn^{-1} + Cn^{-d/(2q)+1} \to 0.
\end{aligned}$$

$\square$

5 Asymptotics of the growth rate for small and large cross-sections

We investigate the behaviour of the asymptotic growth rate $\gamma(v, rK)$ of the volume of the (X, rK)-sausage S_t^{rK} in the limits $r \to 0$ and $r \to \infty$. Recall the stochastic differential equation (1.1) for X and recall the rescaled process $X^{(\sigma)}$ from Section 2. Set $W_t^{(\sigma)} = \sigma^{-1} W_{\sigma^2 t}$. Then $W^{(\sigma)}$ is also a Brownian motion and $X^{(\sigma)}$ satisfies the stochastic differential equation

$$dX_t^{(\sigma)} = dW_t^{(\sigma)} + v^{(\sigma)}(X_t^{(\sigma)})\, dt \tag{5.1}$$

where $v^{(\sigma)}(x) = \sigma v(\sigma x)$. This makes it clear that $X^{(\sigma)} \to W$ as $\sigma \to 0$ weakly on $C([0,\infty), \mathbb{R}^d)$. Recall from Section 2 the fact that $X^{(\sigma)} \to \bar{X}$ as $\sigma \to \infty$, in the same sense, where $\bar{X}$ is a Brownian motion with diffusivity $\bar{a}$.

Take $\sigma = r$. Then

$$S_t^{rK}(X) = rS_{r^{-2}t}^K(X^{(r)})$$

so

$$|S_t^{rK}(X)| = r^d |S_{r^{-2}t}^K(X^{(r)})|.$$

Hence the limit

$$\gamma(v^{(r)}, K) := \lim_{t\to\infty} |S_t^K(X^{(r)})|/t$$

exists and equals $r^{2-d}\gamma(v, rK)$. The weak limits for $X^{(r)}$ as $r \to 0$ or $r \to \infty$ suggest the following result, which however requires further argument because the asymptotic growth rate of the sausage is not a continuous function on $C([0,\infty), \mathbb{R}^d)$. Write $\mathrm{cap}(K)$ for the Newtonian capacity of K and $\mathrm{cap}_{\bar{a}}(K)$ for the capacity of K with respect to the diffusivity matrix $\bar{a}$. Thus

$$\mathrm{cap}_{\bar{a}}(K) = \sqrt{\det \bar{a}}\ \mathrm{cap}(\bar{a}^{-1/2}K).$$

Theorem 5.1 *We have*

$$\lim_{r\to 0} r^{2-d}\gamma(v, rK) = \lim_{r\to 0}\gamma(v^{(r)}, K) = \gamma(0, K) = \mathrm{cap}(K)$$

and

$$\lim_{r\to\infty} r^{2-d}\gamma(v, rK) = \lim_{r\to\infty}\gamma(v^{(r)}, K) = \mathrm{cap}_{\bar{a}}(K).$$

Proof Fix $T \in (0,\infty)$ and write $I(j)$ for the interval $((j-1)T, jT]$. Consider for $1 \leqslant j \leqslant k$ the function $F_{j,k}$ on $C([0,\infty), \mathbb{R}^d)$ defined by

$$F_{j,k}(\gamma) = |S_{I(j)}^K(\gamma) \cap S_{I(k)}^K(\gamma)|/T.$$

Then $F_{j,k}$ is continuous, so

$$\lim_{r\to 0}\mathbb{E}(F_{j,k}(X^{(r)})) = \mathbb{E}(F_{j,k}(W)), \quad \lim_{r\to\infty}\mathbb{E}(F_{j,k}(X^{(r)})) = \mathbb{E}(F_{j,k}(\bar{X})).$$

Choose X_0 so that $\pi(X_0)$ is uniformly distributed. Then by stationarity $\mathbb{E}(F_{j,k}(X^{(r)})) = \mathbb{E}(F_{k-j}(X^{(r)}))$ where $F_j = F_{1,j+1}$. Fix r and write $S_I^K(X^{(r)}) = S_I$. Note that

$$S_{(0,nT]} = S_{I(1)} \cup \cdots \cup S_{I(n)}.$$

So, by inclusion-exclusion, we obtain

$$n\mathbb{E}(F_0(X^{(r)})) - \sum_{j=1}^{n-1}(n-j)\mathbb{E}(F_j(X^{(r)})) \leqslant \mathbb{E}(|S_{(0,nT]}|/T) \leqslant n\mathbb{E}(F_0(X^{(r)})).$$

Divide by n and let $n \to \infty$ to obtain

$$\mathbb{E}(F_0(X^{(r)})) - \sum_{j=1}^{\infty} \mathbb{E}(F_j(X^{(r)})) \leqslant \gamma(v^{(r)}, K) \leqslant \mathbb{E}(F_0(X^{(r)})).$$

Fix $q \in (1, 3/2)$ and $p \in (3, \infty)$ with $p^{-1} + q^{-1} = 1$. By Lemmas 4.1 and 4.2, there is a constant $C(p, v, K) < \infty$ such that, for all r and j,

$$\begin{aligned}\mathbb{E}(F_j(X^{(r)})) &= \mathbb{E}\left(|S_{I(1)} \cap S_{I(j+1)}|/T\right) \\ &\leqslant \mathbb{E}\left(\left||S_{I(1)}|/T\right|^p\right)^{1/p} \mathbb{P}\left(S_{I(1)} \cap S_{I(j+1)} \neq \emptyset\right)^{1/q} \leqslant 2C(j-1)^{-d/2q}.\end{aligned}$$

Given $\varepsilon > 0$, we can choose $J(p, v, K) < \infty$ so that

$$\sum_{j=J+1}^{\infty} \mathbb{E}(F_j(X^{(r)})) \leqslant 2C \sum_{j=J}^{\infty} j^{-d/2q} \leqslant \varepsilon. \tag{5.2}$$

We follow from this point the case $r \to \infty$. The argument for the other limit is the same. Let $r \to \infty$ to obtain

$$\begin{aligned}\mathbb{E}(F_0(\bar{X})) - \sum_{j=1}^{J} \mathbb{E}(F_j(\bar{X})) - \varepsilon &\leqslant \liminf_{r\to\infty} \gamma(v^{(r)}, K) \\ &\leqslant \limsup_{r\to\infty} \gamma(v^{(r)}, K) \leqslant \mathbb{E}(F_0(\bar{X})).\end{aligned} \tag{5.3}$$

It is known that

$$\lim_{T\to\infty} \mathbb{E}(F_0(\bar{X})) = \lim_{T\to\infty} \mathbb{E}(|S_T^K(\bar{X})|/T) = \mathrm{cap}_{\bar{a}}(K).$$

See [9] for the case $\bar{a} = I$. The general case follows by a scaling transformation. Note that, for $j \geqslant 1$,

$$|S^K_{(0,T]}(\bar{X}) \cap S^K_{(jT,(j+1)T]}(\bar{X})| + |S^K_{(0,(j+1)T]}(\bar{X})| \leqslant \sum_{i=1}^{j+1} |S^K_{((i-1)T,iT]}(\bar{X})|.$$

Take expectation, divide by T and let $T \to \infty$ to obtain

$$\begin{aligned}(j+1)\,\mathrm{cap}_{\bar{a}}(K) + \limsup_{T\to\infty} \mathbb{E}|S^K_{(0,T]}(\bar{X}) \cap S^K_{(jT,(j+1)T]}(\bar{X})|/T \\ \leqslant (j+1)\,\mathrm{cap}_{\bar{a}}(K)\end{aligned}$$

which says exactly that

$$\lim_{T\to\infty} \mathbb{E}(F_j(\bar{X})) = 0.$$

Hence the desired limit follows on letting $T \to \infty$ in (5.3). □

6 Homogenization of the advection-diffusion equation in a perforated domain

Our main results are analogues to the homogenization statements (2.3), (2.4) for advection-diffusion in a perforated domain. Recall that v is a $\mathbb{Z}^d$-periodic, Lipschitz, mean-zero, divergence-free vector field on $\mathbb{R}^d$, and K is a compact subset of $\mathbb{R}^d$. The domain $D \subseteq \mathbb{R}^d$ is constructed by removing all the sets $K+p$, where p runs over the set P of atoms of a Poisson random measure μ on $\mathbb{R}^d$ of intensity $\rho = \sigma^{-2}$. Write

$$\bar{a} = \bar{a}(v, K), \quad \bar{\lambda} = \bar{\lambda}(v, K) = \gamma(v, \hat{K}).$$

Theorem 6.1 *Let $f \in L^2(\mathbb{R}^d)$ and $\sigma \in (0, \infty)$ be given. Denote by u the solution*[2] *to the Cauchy problem for*

$$\mathcal{L} = \tfrac{1}{2}\Delta + v(x).\nabla$$

in $[0,\infty) \times D$ with initial data $f(\cdot/\sigma)$, and with Dirichlet conditions on the boundary of D. Set $u^{(\sigma)}(t,x) = u(\sigma^2 t, \sigma x)$. Then

$$\mathbb{E} \int_{\mathbb{R}^d} |u^{(\sigma)}(t,x) - \bar{u}(t,x)|^2 \, dx \to 0$$

as $\sigma \to \infty$, for all $t \geqslant 0$, where $\bar{u}$ is the solution to the Cauchy problem for $\frac{1}{2} \operatorname{div} \bar{a} \nabla - \bar{\lambda}$ in $[0,\infty) \times \mathbb{R}^d$ with initial data f.

Proof Replace t by $\sigma^2 t$, x by σx and f by $f(\cdot/\sigma)$ in (1.2) and (1.3) to obtain

$$\mathbb{E}\left(u^{(\sigma)}(t,x)\right) = \mathbb{E}_{\sigma x}\left(f(X_t^{(\sigma)}) \exp\{-\rho |S_{\sigma^2 t}^{\hat{K}}(X)|\}\right)$$

and

$$\mathbb{E}\left(u^{(\sigma)}(t,x)^2\right) = \mathbb{E}_{\sigma x}\left(f(X_t^{(\sigma)}) f(Y_t^{(\sigma)}) \exp\{-\rho |S_{\sigma^2 t}^{\hat{K}}(X) \cup S_{\sigma^2 t}^{\hat{K}}(Y)|\}\right)$$

where the subscript σx specifies the starting point of X and where Y is an independent copy of X. We omit from now on the superscript $\hat{K}$.

[2] We extend u to a function on $[0,\infty) \times \mathbb{R}^d$ by setting $u(t,x) = 0$ for any $x \notin D$.

Then[3]

$$
\begin{aligned}
&\mathbb{E}\left(|u^{(\sigma)}(t,x)-\bar{u}(t,x)|^2\right)\\
&=\mathbb{E}_{\sigma x}\left(f(X_t^{(\sigma)})f(Y_t^{(\sigma)})e^{-\rho|S_{\sigma^2 t}(X)\cup S_{\sigma^2 t}(Y)|}(1-e^{-\rho|S_{\sigma^2 t}(X)\cap S_{\sigma^2 t}(Y)|})\right)\\
&\quad+\Big(\mathbb{E}_{\sigma x}(f(X_t^{(\sigma)})\{e^{-\rho|S_t(X)|}-e^{-\bar{\lambda}t}\}\\
&\qquad+\left\{\mathbb{E}_{\sigma x}(f(X_t^{(\sigma)}))-\mathbb{E}_x(f(\bar{X}_t))\right\}e^{-\bar{\lambda}t}\Big)^2\\
&\leqslant \mathbb{E}_{\sigma x}\left(f(X_t^{(\sigma)})^2\right)\mathbb{E}_{\sigma x}\left(\rho|S_{\sigma^2 t}(X)\cap S_{\sigma^2 t}(Y)|\right)^{1/2}\\
&\quad+2\mathbb{E}_{\sigma x}\left(f(X_t^{(\sigma)})^2\right)\mathbb{E}_{\sigma x}\left(\rho|S_{\sigma^2 t}(X)|-\bar{\lambda}t|\right)\\
&\qquad+2|u_0^{(\sigma)}(t,x)-\bar{u}_0(t,x)|^2
\end{aligned}\tag{6.1}
$$

where $u_0^{(\sigma)}$ and $\bar{u}_0$ denote the corresponding solutions to the Cauchy problem with initial data f in the whole space, and we used Cauchy–Schwarz and $(a+b)^2 \leqslant 2a^2+2b^2$ and $|e^{-a}-e^{-b}|^2 \leqslant |b-a|$ to obtain the inequality. Now

$$\int_{\mathbb{R}^d}\mathbb{E}_{\sigma x}\left(f(X_t^{(\sigma)})^2\right)dx=\int_{\mathbb{R}^d}|f(x)|^2\,dx<\infty$$

because dx is stationary for $X^{(\sigma)}$ and, by (2.3), as $\sigma\to\infty$

$$\int_{\mathbb{R}^d}|u_0^{(\sigma)}(t,x)-\bar{u}_0(t,x)|^2\,dx\to 0.$$

So, using Lemma 4.3, on integrating (6.1) over $\mathbb{R}^d$ and letting $\sigma\to\infty$, we conclude that the right-hand side tends to 0, proving the theorem. □

Theorem 6.2 *Let $x\in\mathbb{R}^d$ and $\sigma\in(0,\infty)$ be given. Let X be an $\mathcal{L}$-diffusion in $\mathbb{R}^d$ starting from σx and set*

$$T=\inf\{t\geqslant 0: X_t\in K+P\}.$$

Set $X_t^{(\sigma)}=\sigma^{-1}X_{\sigma^2 t}$ and $T^{(\sigma)}=\sigma^{-2}T$. Write $\bar{X}$ for a Brownian motion in $\mathbb{R}^d$ with diffusivity $\bar{a}$ starting from x, and write $\bar{T}$ for an exponential random variable of parameter $\bar{\lambda}$, independent of $\bar{X}$. Then, as $\sigma\to\infty$,

$$(X^{(\sigma)},T^{(\sigma)})\to(\bar{X},\bar{T}),\quad \textit{weakly on } C([0,\infty),\mathbb{R}^d)\times[0,\infty).$$

Proof Write S_t for the $(X,\hat{K})$-sausage. Fix a bounded continuous function F on $C([0,\infty),\mathbb{R}^d)$ and fix $t>0$. Then

$$\mathbb{E}\left(F(X^{(\sigma)})1_{\{T^{(\sigma)}>t\}}\right)=\mathbb{E}\left(F(X^{(\sigma)})\exp\{-\rho|S_{\sigma^2 t}|\}\right)$$

[3] This is an instance of the formula $\mathbb{E}(|X-a|^2)=\mathrm{var}(X)+(\mathbb{E}(X)-a)^2$.

and

$$\mathbb{E}\left(F(\bar{X})1_{\{\bar{T}>t\}}\right) = \mathbb{E}\left(F(\bar{X})e^{-\bar{\lambda}t}\right)$$

so

$$\begin{aligned}&\left|\mathbb{E}\left(F(X^{(\sigma)})1_{\{T^{(\sigma)}>t\}}\right) - \mathbb{E}\left(F(\bar{X})1_{\{\bar{T}>t\}}\right)\right| \\ &\qquad \leqslant \|F\|_\infty \, \mathbb{E}_{\sigma x}|\rho|S_{\sigma^2 t}| - \bar{\lambda}t| + |\mathbb{E}(F(X^{(\sigma)}) - \mathbb{E}(F(\bar{X}))|e^{-\bar{\lambda}t}.\end{aligned}$$

On letting $\sigma \to \infty$, the first term on the right tends to 0 by Lemma 4.3 and the second term tends to 0 by (2.4), so the left hand side also tends to 0, proving the theorem. □

7 The case of diffusivity $\varepsilon^2 I$

In this section and the next we fix $\varepsilon \in (0,\infty)$ and consider the more general case of the operator

$$\mathcal{L} = \tfrac{1}{2}\varepsilon^2\Delta + v(x).\nabla.$$

The following statements follow from the corresponding statements above for the case $\varepsilon = 1$ by scaling. Fix $x \in \mathbb{R}^d$ and let X be an $\mathcal{L}$-diffusion in $\mathbb{R}^d$ starting from x. Then

$$|S_t^K(X)|/t \to \gamma(\varepsilon, v, K), \quad \mathbb{P}_x\text{-almost surely}$$

as $t \to \infty$, where

$$\gamma(\varepsilon, v, K) = \varepsilon^2\gamma(\varepsilon^{-2}v, K).$$

Moreover, setting $v^{(r)}(x) = rv(rx)$, as above, we have

$$\gamma(\varepsilon, v^{(r)}, K) \to \operatorname{cap}_{\varepsilon^2 I}(K) = \varepsilon^2 \operatorname{cap}(K), \quad \text{as } r \to 0$$

and

$$\gamma(\varepsilon, v^{(r)}, K) \to \operatorname{cap}_{\bar{a}(\varepsilon,v)}(K), \quad \text{as } r \to \infty$$

where

$$\bar{a}(\varepsilon, v) = \varepsilon^2\bar{a}(\varepsilon^{-2}v).$$

Fix $\sigma \in (0,\infty)$ and suppose now that X starts at σx. Define as above $T = \inf\{t \geqslant 0 : X_t \in K + P\}$ and write $X_t^{(\sigma)} = \sigma^{-1}X_{\sigma^2 t}$ and $T^{(\sigma)} = \sigma^{-2}T$. Then, as $\sigma \to \infty$,

$$(X^{(\sigma)}, T^{(\sigma)}) \to (\bar{X}, \bar{T}), \quad \text{weakly on } C([0,\infty),\mathbb{R}^d) \times [0,\infty)$$

where $\bar{X}$ is a Brownian motion in $\mathbb{R}^d$ of diffusivity $\bar{a}(\varepsilon, v)$ starting from x, and where $\bar{T}$ is an exponential random variable independent of $\bar{X}$, of parameter $\bar{\lambda}(\varepsilon, v, K) = \gamma(\varepsilon, v, \hat{K})$.

8 Monte Carlo computation of the asymptotic growth rate

Let X be as in the preceding section. Fix $T \in (0, \infty)$. The following algorithm may be used to estimate numerically the volume of the (X, K)-sausage $S_T = S_T^K(X)$. The algorithm is determined by the choice of three parameters N, m, $J \in \mathbb{N}$.

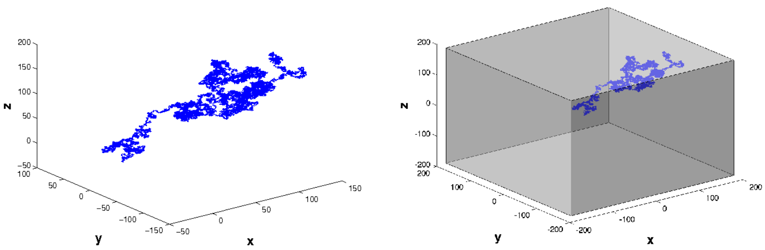
- **Step 1:** *Compute an Euler–Maruyama solution* $(X_{n\Delta t} : n = 0, 1, \ldots, N)$ *to the stochastic differential equation*

$$dX_t = \varepsilon dW_t + v(X_t)\, dt, \quad X_0 = x \tag{8.1}$$

  up to the final time $T = N\Delta t$ (Figure 8.1a).
- **Step 2:** *Calculate*

$$R_K = \max_{y \in K,\, 1 \leqslant k \leqslant d} |y^k|, \quad R_{X,T} = \max_{1 \leqslant n \leqslant N,\, 1 \leqslant k \leqslant d} |X^k_{n\Delta t} - x^k|.$$

  We approximate S_T by $S_T^{(N)} = \cup_{0 \leqslant n \leqslant N}(K + X_{n\Delta t})$. Note that $S_T^{(N)}$ is contained in the cube with side-length $L = 2(R_K + R_{X,T})$ centred at x (Figure 8.1b).

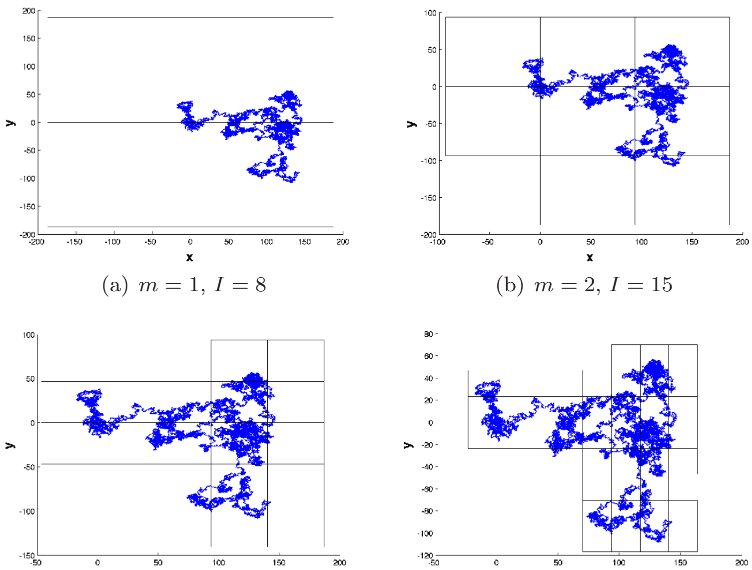


(a) Step 1 (b) Step 2

Figure 8.1 First two steps of the algorithm

- **Step 3:** *Subdivide the cube of side-length L centred at x into 2^d sub-cubes of side-length $L/2$ and check which of them have non-empty intersection with $S_T^{(N)}$. Discard any sub-cubes with empty intersection.*

Repeat the division and discarding procedure in each of the remaining sub-cubes (Figure 8.2) iteratively to obtain I sub-cubes of side-length $L/2^m$, centred at $y_1, \ldots, y_I$ say, whose union contains $S_T^{(N)}$.



(a) $m = 1$, $I = 8$ (b) $m = 2$, $I = 15$

(c) $m = 3$, $I = 40$ (d) $m = 4$, $I = 111$

Figure 8.2 $x - y$ projection of the path and the sub-cubes for different values of m

- **Step 4:** *Generate uniform random variables $U_1, \ldots, U_J$ in $[-1/2, 1/2]^d$ and estimate $V = |S_T|$ by*

$$\hat{V} = J^{-1} 2^{-md} L^d \sum_{i=1}^{I} \sum_{j=1}^{J} \left(1 - \prod_{n=0}^{N} 1_{A_n(i,j)} \right)$$

where $A_n(i,j) = \{y_i + 2^{-m} L U_j \notin K + X_{n\Delta t}\}$.

The algorithm was tested in the case $d = 3$. We took $\varepsilon = 0.25$ and took v to be the Taylor–Green vector field in the first two co-ordinate directions; thus

$$v(x) = (-\sin x_1 \cos x_2, \, \cos x_1 \sin x_2, \, 0)^T.$$

We applied the algorithm to $X^{(r)}$, which has drift vector field $v^{(r)}(x) = rv(rx)$, for a range of choices of $r \in (0, \infty)$. We took K to be the Eu-

clidean unit ball B and computed $|S_T^B(X^{(r)})|$ for $T = 10^4$, using parameter values $N = 10^6$, $m = 4$ and $J = 10^4$. The numerical method used to solve (8.1) was taken from [14]. The values $|S_T^B(X^{(r)})|/T$ were taken as estimates of the asymptotic volume growth rate $\gamma(0.25, v^{(r)}, B)$. These are displayed in Figure 8.3.

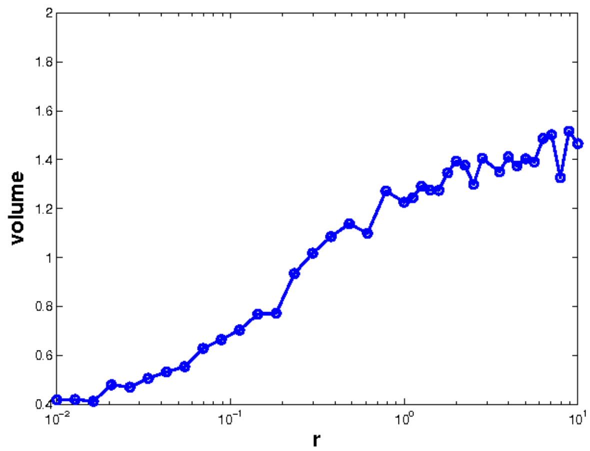


Figure 8.3 Growth rate of the sausage for different values of r

In Section 7 we stated the following theoretical limit, deduced from Theorem 5.1:

$$\lim_{r\to 0} \gamma(\varepsilon, v^{(r)}, B) = \varepsilon^2 \operatorname{cap}(B) = 2\pi\varepsilon^2 = 0.3927. \tag{8.2}$$

This is consistent with the computed values of $\gamma(0.25, v^{(r)}, B)$ when r is small.

It is known [10] that $\bar{a}(\varepsilon, v)$ has the form

$$\bar{a}(\varepsilon, v) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \varepsilon^2/2 \end{pmatrix}$$

for some $\alpha = \alpha(\varepsilon, v)$, which can be computed using Monte Carlo simulations. In [16], this was carried out for $\varepsilon = 0.25$ up to a final time $T = 10^4$, using a time step $\Delta t = 10^{-2}$, again using a numerical method from [14] to solve (8.1). The value $\alpha(0.25, v) = 0.0942$ was obtained as the sample average over 10^4 realizations of $|X_T^1|^2/T$. Using this value, we simulated $\bar{X}$ and used the volume algorithm to compute $|S_T^B(\bar{X})|/T$

as an approximation to $\text{cap}_{\bar{a}(0.25,v)}(B)$, obtaining the value 1.4587. We showed theoretically that

$$\lim_{r\to\infty} \gamma(\varepsilon, v^{(r)}, B) = \text{cap}_{\bar{a}(\varepsilon,v)}(B).$$

The computed value for $\text{cap}_{\bar{a}(0.25,v)}(B)$ is consistent with the computed values of $\gamma(0.25, v^{(r)}, B)$ for large r.

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